

Quantization of a generally covariant gauge system with two super Hamiltonian constraints

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Abstract

The Becci-Rouet-Stora-Tyutin (BRST) operator quantization of a finite-dimensional gauge system featuring two quadratic super Hamiltonian and m linear supermomentum constraints is studied as a model for quantizing generally covariant gauge theories. The proposed model “completely” mimics the constraint algebra of general relativity. The Dirac constraint operators are identified by realizing the BRST generator of the system as a Hermitian nilpotent operator, and a physical inner product is introduced to complete a consistent quantization procedure.

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I. INTRODUCTION

Generally covariant theories such as Einstein's theory of gravitation have the peculiar property of featuring a Hamiltonian that is constrained to vanish. This constraint is associated with the invariance of the action under reparametrizations. In the case of general relativity, the Arnowitt-Deser-Misner (ADM) Hamiltonian is the sum of four constraint functions, three of them are the supermomenta \mathcal{H}_a (linear and homogeneous functions of the field momenta) and the other one is the super-Hamiltonian \mathcal{H} (a quadratic function of the field momenta). The supermomenta generate the change of the field under diffeomorphisms on the spatial hypersurfaces Σ of the foliated space-time manifold, and express the invariance of the action under these transformations. The super-Hamiltonian generates the evolution of the field under displacements that are normal to the hypersurface Σ ; this evolution is nothing but a physically irrelevant reparametrization of the dynamical trajectory of the system. Actually there are four constraint functions in each point of Σ . So the algebra of constraints is rather complicated, because one should evaluate the Poisson brackets between constraints at different points. In particular, the Poisson brackets between two super-Hamiltonian take part in the algebra of constraints. This algebra was calculated by Dirac [1]:

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^a(x) \delta_{,a}(x, x') + \mathcal{H}^a(x') \delta_{,a}(x, x'), \quad (1.1)$$

$$\{\mathcal{H}_a(x), \mathcal{H}(x')\} = \mathcal{H}(x) \delta_{,a}(x, x'), \quad (1.2)$$

$$\{\mathcal{H}_a(x), \mathcal{H}_b(x')\} = \mathcal{H}_b(x) \delta_{,a}(x, x') + \mathcal{H}_a(x') \delta_{,b}(x, x'), \quad (1.3)$$

and his first class character guarantees that the dynamical trajectories are consistent with the constraints

$$\mathcal{H} = 0, \quad \mathcal{H}_a = 0. \quad (1.4)$$

According to Dirac, the quantization of a constrained system requires a factor ordering able to preserve the algebra of constraints at the level of operators (absence of anomalies). In the case of general relativity this issue remains unsolved [2,3]. In order to avoid the regularization of operators, it is a common practice to study this question in finite-dimensional

systems featuring constraints that resemble the algebra of general relativity. Although these types of systems have been widely studied (see, for example, Refs. [4–7]), it has been pointed out by Montesinos et al. [8] that an important feature of the algebra (1.1)-(1.3) has not been sufficiently examined: usually people does not deal with Eq. (1.1) but only with (1.2),(1.3), i.e. frequently only one super-Hamiltonian is included (however, models with several *commuting* Hamiltonian constraints were considered in Refs. [9–11]). Actually, Montesinos et al. [8] solved a simple model with two euclidean flat super-Hamiltonians plus a single supermomenta satisfying only Eqs. (1.1)-(1.2) with constant structure coefficients; thanks to the simple structure of the constraints, the ordering problem is trivially solved in this case. Our aim is to quantize a more complex system in order to deal with a non trivial ordering of constraint operators that mimic the complete algebra (1.1)-(1.3).

It is well known that, besides the Dirac method, the Becchi-Rouet-Stora-Tyutin (BRST) formalism is a powerful tool to quantize a first class constrained system [12] (see also Ref. [13] for an account of the Batalin-Fradkin-Vilkovisky formalism). In recent works [5–7], we have taken advantage of its strength for obtaining the consistently ordered constraint operators belonging to generally covariant systems including only one super-Hamiltonian constraint. Here, we will show that the tools there developed are also useful in the treatment of a nontrivial system featuring *two* super-Hamiltonian constraints and *m* supermomenta constraints.

We will start by defining the model. Then, the system will be quantized within the framework of the BRST formalism, where the nilpotency of the BRST generator must be proven in order to guarantee an anomaly free quantization. The Dirac constraint operators will be identified from the nilpotent BRST generator. In spite of this quantization will be performed for a system featuring an algebra with constant structure functions, the results will be extended to more general algebras by means of a unitary transformation. Finally, a physical inner product will be introduced to complete a consistent quantization procedure.

II. THE MODEL

Let us consider a system described by $4n$ canonical coordinates (q^i, p_i) , subjected to *two* super-Hamiltonian constraints

$$\mathcal{H}_1 = \frac{1}{2} g^{i_1 j_1}(q^{k_1}) p_{i_1} p_{j_1} + v_1(q^{k_2}), \quad (2.1)$$

$$\mathcal{H}_2 = \frac{1}{2} g^{i_2 j_2}(q^{k_2}) p_{i_2} p_{j_2} + v_2(q^{k_1}), \quad (2.2)$$

where $i_1 = 1, \dots, n$ and $i_2 = n + 1, \dots, 2n$.

The metrics $g^{i_1 j_1}$ and $g^{i_2 j_2}$ are indefinite and non degenerated, and depend on the q^{i_1} 's and the q^{i_2} s respectively. On the contrary, the potentials exhibit an opposite functional dependence: $v_1 = v_1(q^{i_2})$ and $v_2 = v_2(q^{i_1})$.

In addition, the system is also subjected to m linearly independent supermomentum constraints

$$\mathcal{H}_a = \xi_a^i p_i, \quad a = 3, \dots, m + 2; \quad i = (i_1, i_2), \quad (2.3)$$

where

$$\xi_a^{i_1} = \xi_a^{i_1}(q^{k_1}), \quad \xi_a^{i_2} = \xi_a^{i_2}(q^{k_2}). \quad (2.4)$$

The special way the geometrical objects in the constraint functions depend on the coordinates, has been chosen for obtaining a constraint algebra which mimics Eqs. (1.1)-(1.3):

$$\{\mathcal{H}_1, \mathcal{H}_2\} = c_{12}^a \mathcal{H}_a, \quad (2.5)$$

$$\{\mathcal{H}_1, \mathcal{H}_a\} = c_{1a}^1 \mathcal{H}_1, \quad (2.6)$$

$$\{\mathcal{H}_2, \mathcal{H}_a\} = c_{2a}^2 \mathcal{H}_2, \quad (2.7)$$

$$\{\mathcal{H}_a, \mathcal{H}_b\} = C_{ab}^c \mathcal{H}_c. \quad (2.8)$$

It should be noticed that this system describes two *interacting* particles. In fact, it is not possible to decouple the system in two subsystems described by (q^{i_1}, p_{i_1}) and (q^{i_2}, p_{i_2}) . As long as we know, models of this class in riemannian manifolds have not been studied yet in the literature.

We will start by quantizing an algebra with constant structure functions. Later, by taking into account the scaling properties of the super-Hamiltonians, we will extend the results to some algebras with non constant structure functions.

In order that the constraints (2.1), (2.2), and (2.3) effectively satisfy the algebra (2.5)-(2.8), metrics, vectors and potentials must fulfill certain relations. By substituting the constraints in Eq. (2.5) one obtains

$$g^{i_2 k_2} v_{1, k_2} p_{i_2} - g^{i_1 k_1} v_{2, k_1} p_{i_1} = c_{12}^a (\xi_a^{i_1} p_{i_1} + \xi_a^{i_2} p_{i_2}), \quad (2.9)$$

then,¹

$$c_{12}^a \xi_a^{i_2} = g^{i_2 k_2} v_{1, k_2} \quad (2.10)$$

and

$$c_{12}^a \xi_a^{i_1} = -g^{i_1 k_1} v_{2, k_1}. \quad (2.11)$$

The substitution of the constraints in Eq. (2.6) yields

$$\frac{1}{2} (g_{, k_1}^{i_1 j_1} \xi_a^{k_1} - 2g^{i_1 k_1} \xi_{a, k_1}^{j_1}) p_{i_1} p_{j_1} + v_{1, k_2} \xi_a^{k_2} = c_{1a}^1 (g^{i_1 j_1} p_{i_1} p_{j_1} + v_1) \quad (2.12)$$

then

$$g_{, k_1}^{i_1 j_1} \xi_a^{k_1} - 2g^{i_1 k_1} \xi_{a, k_1}^{j_1} = c_{1a}^1 g^{i_1 j_1} \quad (2.13)$$

and

¹It must be noticed that Eqs. (2.10) and (2.11) do not force c_{12}^a to be a constant; they only imply that $c_{12}^a = {}^{(1)}c_{12}^a(q^{i_1}) - {}^{(2)}c_{12}^a(q^{i_2})$, where ${}^{(1)}c_{12}^a \xi_a^{i_1} = 0$ and ${}^{(2)}c_{12}^a \xi_a^{i_2} = 0$.

$$\xi_a^{k_2} v_{1,k_2} = c_{1a}^1 v_1. \quad (2.14)$$

In geometrical language, the relations (2.13) and (2.14) read

$$\mathcal{L}_{\vec{\xi}_a} \bar{g}_1 = c_{1a}^1 \bar{g}_1, \quad (2.15)$$

$$\mathcal{L}_{\vec{\xi}_a} v_1 = c_{1a}^1 v_1. \quad (2.16)$$

Analogously, the substitution of the constraints in Eq. (2.7) leads to

$$\frac{1}{2}(g_{,k_2}^{i_2 j_2} \xi_a^{k_2} - 2g^{i_2 k_2} \xi_{a,k_2}^{j_2}) p_{i_2} p_{j_2} + v_{2,k_1} \xi_a^{k_1} = c_{2a}^2 (g^{i_2 j_2} p_{i_2} p_{j_2} + v_2) \quad (2.17)$$

which means

$$g_{,k_2}^{i_2 j_2} \xi_a^{k_2} - 2g^{i_2 k_2} \xi_{a,k_2}^{j_2} = c_{2a}^2 g^{i_2 j_2} \quad (2.18)$$

and

$$\xi_a^{k_1} v_{2,k_1} = c_{2a}^2 v_2 \quad (2.19)$$

or, in geometrical language,

$$\mathcal{L}_{\vec{\xi}_a} \bar{g}_2 = c_{2a}^2 \bar{g}_2, \quad (2.20)$$

$$\mathcal{L}_{\vec{\xi}_a} v_2 = c_{2a}^2 v_2. \quad (2.21)$$

Thus, the fulfillment of the algebra means that the supermomenta are conformal Killing vectors of the super-Hamiltonians.

Finally, by substituting the supermomenta in Eq. (2.8) one obtains that ²

$$(\xi_a^{i_1} \xi_{b,i_1}^{j_1} - \xi_b^{i_1} \xi_{a,i_1}^{j_1}) p_{j_1} + (\xi_a^{i_2} \xi_{b,i_2}^{j_2} - \xi_b^{i_2} \xi_{a,i_2}^{j_2}) p_{j_2} = C_{ab}^c (\xi_c^{i_1} p_{i_1} + \xi_c^{i_2} p_{i_2}). \quad (2.22)$$

²Eq. (2.22) does not imply that the C_{ab}^c s are constant, it only imposes the decomposition: $C_{ab}^c = {}^{(1)}C_{ab}^c(q^{i_1}) - {}^{(2)}C_{ab}^c(q^{i_2})$ where ${}^{(1)}C_{ab}^c \xi_c^{i_1} = 0$ and ${}^{(2)}C_{ab}^c \xi_c^{i_2} = 0$.

III. THE BRST FORMALISM: CLASSICAL AND QUANTUM GENERATORS

Our aim is to find the constraint operators that satisfy the algebra at the quantum level. The appropriate factor ordering can be found within the framework of the BRST formalism. For this, the original phase space is extended by including a canonically conjugate pair of fermionic ghosts (η^a, \mathcal{P}_a) for each constraint function. The central object is the BRST generator, a fermionic function $\Omega = \Omega(q^i, p_j, \eta^a, \mathcal{P}_b)$ that captures all of identities satisfied by the set of first class constraints in a unique identity

$$\{\Omega, \Omega\} = 0. \quad (3.1)$$

The existence of Ω is guaranteed at the classical level, and Ω is unique up to canonical transformations in the extended phase space. It can be built by means of a recursive method [12]. In the present case the result is (a closed algebras has “rank” equal to 1)

$$\Omega = \eta^1 \mathcal{H}_1 + \eta^2 \mathcal{H}_2 + \eta^a \mathcal{H}_a + \eta^1 \eta^2 c_{12}^a \mathcal{P}_a + \eta^1 \eta^a c_{1a}^1 \mathcal{P}_1 + \eta^2 \eta^a c_{2a}^2 \mathcal{P}_2 + \frac{1}{2} \eta^a \eta^b C_{ab}^c \mathcal{P}_c. \quad (3.2)$$

In order to quantize the extended system, the classical BRST generator must be realized as a Hermitian operator. The theory is free from BRST anomalies, if a Hermitian realization of Ω can be found such that the classical property (3.1) becomes

$$[\hat{\Omega}, \hat{\Omega}] = 2\hat{\Omega}^2 = 0, \quad (3.3)$$

i.e., $\hat{\Omega}$ must be nilpotent. The BRST physical quantum states belong to the set of equivalence classes of BRST-closed states ($\hat{\Omega}\psi = 0$) moduli BRST-exact ones ($\psi = \hat{\Omega}\chi$) (quantum BRST cohomology).

In order to get the Hermitian and nilpotent operator $\hat{\Omega}$ is helpful to write Ω with the canonical and ghost momenta on an equal footing. So, let us adopt the notation

$$\eta^{C_s} = (q^i, \eta^\beta), \quad \mathcal{P}_{C_s} = (p_i, \mathcal{P}_\beta), \quad (3.4)$$

where $s = -1, 0$, and $\beta = (A; a) = (1, 2; 3, \dots, m+2)$. The index s distinguishes original variables from the added ghost variables.

Also, Ω is a sum of a term quadratic in the momenta

$$\Omega^{quad} = \eta^A \mathcal{H}_A \equiv \frac{1}{2} \sum_{r,s=-1}^0 \Omega^{A_r B_s} \mathcal{P}_{A_r} \mathcal{P}_{B_s} + \eta^A v_A, \quad (3.5)$$

plus another term linear in the momenta

$$\Omega^{linear} = \eta^a \mathcal{H}_a + \eta^1 \eta^2 c_{12}^a \mathcal{P}_a + \eta^1 \eta^a c_{1a}^1 \mathcal{P}_1 + \eta^2 \eta^a c_{2a}^2 \mathcal{P}_2 + \frac{1}{2} \eta^a \eta^b C_{ab}^c \mathcal{P}_c \equiv \sum_{s=-1}^0 \Omega^{c_s} \mathcal{P}_{c_s}. \quad (3.6)$$

In general, it is a difficult step to get the Hermitian nilpotent BRST operator from its classical counterpart because a general method is not available (when not unrealizable: at the quantum level its mere existence is not guaranteed). Nevertheless, let us begin by proposing for the linear term the operator [5,6]

$$\hat{\Omega}^{linear} = \sum_{s=-1}^0 f^{\frac{1}{2}} \hat{\Omega}^{c_s} \hat{\mathcal{P}}_{c_s} f^{-\frac{1}{2}} \quad (3.7)$$

where $f = f(q^i)$ depends on all of original coordinates.

$\hat{\Omega}^{linear}$ will be Hermitian if f satisfies

$$C_{a\beta}^\beta = f^{-1} (f \xi_a^i)_{,i}, \quad (3.8)$$

where $C_{a\beta}^\beta = C_{ab}^b + c_{a1}^1 + c_{a2}^2$ (please remember that β runs over all constraint labels). In Eq. (3.8), f behaves as a volume in the gauge orbit of the supermomenta (see Ref. [5]).

The quadratic term can be caught in the Hermitian ordering:

$$\hat{\Omega}^{quad} = \frac{1}{2} \sum_{r,s=-1}^0 f^{-\frac{1}{2}} \hat{\mathcal{P}}_{A_r} f \Omega^{A_r B_s} \hat{\mathcal{P}}_{B_s} f^{-\frac{1}{2}} + \hat{\eta}^A v_A. \quad (3.9)$$

Finally, the proposed Hermitian BRST operator can be rearranged in the $\hat{\eta} - \hat{\mathcal{P}}$ order by repeatedly using the ghost (anti)commutation relations. After this procedure is completed, the classical structure of Eq. (3.2) will be reproduced at the quantum level, although it will be free of anomalies only if $\hat{\Omega}$ is nilpotent [12]

$$\hat{\Omega} = \hat{\eta}^1 \hat{\mathcal{H}}_1 + \hat{\eta}^2 \hat{\mathcal{H}}_2 + \hat{\eta}^a \hat{\mathcal{H}}_a + \hat{\eta}^1 \hat{\eta}^2 c_{12}^a \hat{\mathcal{P}}_a + \hat{\eta}^1 \hat{\eta}^a c_{1a}^1 \hat{\mathcal{P}}_1 + \hat{\eta}^2 \hat{\eta}^a c_{2a}^2 \hat{\mathcal{P}}_2 + \frac{1}{2} \hat{\eta}^a \hat{\eta}^b C_{ab}^c \hat{\mathcal{P}}_c. \quad (3.10)$$

In this case, the result is

$$\hat{\mathcal{H}}_1 = \frac{1}{2}f^{-\frac{1}{2}}\hat{p}_{i_1}g^{i_1j_1}f\hat{p}_{j_1}f^{-\frac{1}{2}} + v_1, \quad (3.11)$$

$$\hat{\mathcal{H}}_2 = \frac{1}{2}f^{-\frac{1}{2}}\hat{p}_{i_2}g^{i_2j_2}f\hat{p}_{j_2}f^{-\frac{1}{2}} + v_2, \quad (3.12)$$

$$\hat{\mathcal{H}}_a = f^{\frac{1}{2}}\xi_a^i\hat{p}_if^{-\frac{1}{2}}. \quad (3.13)$$

We still need to demand the nilpotency of the proposed $\hat{\Omega}$. We expect additional conditions over $f(q^i)$ since Eq. (3.8) does not completely fix it.

Proof of $\hat{\Omega}^2 = 0$. The proof is done by explicit calculation. Here we give an abridged demonstration; the full version can be found in Ref. [14]. After the removing of the terms that cancel out due to the Jacobi identities such as

$$\begin{aligned} & \{\{\mathcal{H}_1, \mathcal{H}_2\}, \mathcal{H}_a\} + \{\{\mathcal{H}_2, \mathcal{H}_a\}, \mathcal{H}_1\} + \{\{\mathcal{H}_a, \mathcal{H}_1\}, \mathcal{H}_2\} = 0 \\ \implies & c_{12}^b C_{ba}^c - c_{12}^c (c_{1a}^1 + c_{2a}^2) = 0, \end{aligned} \quad (3.14)$$

or that are identically null $[(\eta^a)^2 \equiv 0, \text{etc.}]$, one finally gets

$$\begin{aligned} [\hat{\Omega}, \hat{\Omega}] &= 2\hat{\eta}^1\hat{\eta}^2([\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2] - ic_{12}^b\hat{\mathcal{H}}_b) \\ &+ 2\hat{\eta}^1\hat{\eta}^a([\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_a] - ic_{1a}^1\hat{\mathcal{H}}_1) \\ &+ 2\hat{\eta}^2\hat{\eta}^a([\hat{\mathcal{H}}_2, \hat{\mathcal{H}}_a] - ic_{2a}^2\hat{\mathcal{H}}_2). \end{aligned} \quad (3.15)$$

Therefore the nilpotency is apparent whenever the constraint operators $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_2$, and $\hat{\mathcal{H}}_a$ realize the first class constraint algebra at the quantum level:

$$[\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2] = ic_{12}^a\hat{\mathcal{H}}_a, \quad (3.16)$$

$$[\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_a] = ic_{Aa}^{(A)}\hat{\mathcal{H}}_{(A)} \quad (3.17)$$

(there is no sum over label A). Then, let us calculate explicitly Eq. (3.16):

$$\begin{aligned} [\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2] &= [\frac{1}{2}f^{-\frac{1}{2}}\hat{p}_{i_1}g^{i_1j_1}f\hat{p}_{j_1}f^{-\frac{1}{2}} + v_1, \frac{1}{2}f^{-\frac{1}{2}}\hat{p}_{i_2}g^{i_2j_2}f\hat{p}_{j_2}f^{-\frac{1}{2}} + v_2] \\ &= f^{\frac{1}{2}}[\frac{1}{2}f^{-1}\hat{p}_{i_1}g^{i_1j_1}f\hat{p}_{j_1} + v_1, \frac{1}{2}f^{-1}\hat{p}_{i_2}g^{i_2j_2}f\hat{p}_{j_2} + v_2]f^{-\frac{1}{2}} \\ &= f^{\frac{1}{2}}[\frac{1}{2}g^{i_1j_1}\hat{p}_{i_1}\hat{p}_{j_1} - \frac{i}{2}f^{-1}(g^{i_1j_1}f)_{,i_1}\hat{p}_{j_1} + v_1, \\ &\quad , \frac{1}{2}g^{i_2j_2}\hat{p}_{i_2}\hat{p}_{j_2} - \frac{i}{2}f^{-1}(g^{i_2j_2}f)_{,i_2}\hat{p}_{j_2} + v_2]f^{-\frac{1}{2}}. \end{aligned} \quad (3.18)$$

In order to satisfy Eq. (3.16) the commutator should not contain quadratic terms in the momenta, so

$$[-\frac{i}{2}f^{-1}(g^{i_1j_1}f)_{,i_1}\hat{p}_{j_1}, \frac{1}{2}g^{i_2j_2}\hat{p}_{i_2}\hat{p}_{j_2}] \quad \text{and} \quad [-\frac{i}{2}f^{-1}(g^{i_2j_2}f)_{,i_2}\hat{p}_{j_2}, \frac{1}{2}g^{i_1j_1}\hat{p}_{i_1}\hat{p}_{j_1}]$$

should be zero. This can be achieved by demanding that $f(q^i)$ factorizes as

$$f(q^i) = f_1(q^{i_1})f_2(q^{i_2}). \quad (3.19)$$

It should be remarked that the additional condition (3.19) is fully compatible with (3.8).

As a consequence of the property (3.19), the term

$$[-\frac{i}{2}f^{-1}(g^{i_1j_1}f)_{,i_1}\hat{p}_{j_1}, -\frac{i}{2}f^{-1}(g^{i_2j_2}f)_{,i_2}\hat{p}_{j_2}]$$

is also zero.

Then it remains

$$[\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2] = i c_{12}^a f^{\frac{1}{2}} \hat{\mathcal{H}}_a f^{-\frac{1}{2}} - f_1^{-1}(g^{i_1j_1}v_{2,j_1}f_1)_{,i_1} + f_2^{-1}(g^{i_2j_2}v_{1,j_2}f_2)_{,i_2}, \quad (3.20)$$

where we used the classical relation (2.9) in order to rebuilt $\hat{\mathcal{H}}_a$. The last two terms in Eq. (3.20) can be rewritten by using the classical relations (2.10) and (2.11)

$$f_1^{-1}(c_{12}^a \xi_a^{i_1} f_1)_{,i_1} + f_2^{-1}(c_{12}^a \xi_a^{i_2} f_2)_{,i_2} = c_{12}^a f^{-1}(\xi_a^i f)_{,i} = c_{12}^a (C_{ab}^b + c_{a1}^1 + c_{a2}^2), \quad (3.21)$$

to cancel out as a consequence of the Jacobi identity (3.14).

To complete the proof, let us now evaluate the Eq. (3.17):

$$\begin{aligned} [\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_a] &= [\frac{1}{2}f^{-\frac{1}{2}}\hat{p}_{i_A}g^{i_Aj_A}f\hat{p}_{j_A}f^{-\frac{1}{2}} + v_A, f^{\frac{1}{2}}\xi_a^k\hat{p}_kf^{-\frac{1}{2}}] \\ &= \frac{1}{2}f^{-\frac{1}{2}}(\hat{p}_{i_A}g^{i_Aj_A}f\hat{p}_{j_A}\xi_a^k\hat{p}_k - \xi_a^k\hat{p}_k\hat{p}_{i_A}g^{i_Aj_A}f\hat{p}_{j_A})f^{-\frac{1}{2}} + i\xi_a^k v_{A,k} \\ &= \frac{1}{2}f^{-\frac{1}{2}}\left(i\hat{p}_{i_A}(g_{,k_A}^{i_Aj_A}\xi_a^{k_A} - 2g^{i_Ak_A}\xi_{a,k_A}^{j_A})\hat{p}_{j_A} + fg^{i_Aj_A}[\xi_a^k(\ln f)_{,k}]_{,i_A} + \xi_{a,i_Ak}^k\hat{p}_{j_A}\right)f^{-\frac{1}{2}} \\ &\quad + i\xi_a^k v_{A,k}. \end{aligned} \quad (3.22)$$

After using Eqs. (2.13),(2.14) or (2.18), (2.19) one obtains

$$\begin{aligned}
[\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_a] &= i c_{Aa}^A \frac{1}{2} f^{-\frac{1}{2}} \hat{p}_{i_A} g^{i_A j_A} f \hat{p}_{j_A} f^{-\frac{1}{2}} + i \xi_a^k v_{A,k} \\
&\quad + \frac{1}{2} f^{-\frac{1}{2}} \left(f g^{i_A j_A} [\xi_a^k (ln f)_{,k}]_{,i_A} + \xi_{a,i_A k}^k \hat{p}_{j_A} \right) f^{-\frac{1}{2}}.
\end{aligned} \tag{3.23}$$

The first two terms in (3.23) are $i c_{Aa}^{(A)} \hat{\mathcal{H}}_{(A)}$ (there no sum over label A). Then we have to see whether the last term is zero. Actually,

$$[\xi_a^k (ln f)_{,k}]_{,i_A} + \xi_{a,i_A k}^k = [\xi_a^k (ln f)_{,k} + \xi_{a,k}^k]_{,i_A} = [f^{-1} (\xi_a^k f)_{,k}]_{,i_A} = C_{a\beta, i_A}^\beta = 0. \tag{3.24}$$

Thus the demonstration is completed. $\hat{\Omega}$ is Hermitian thanks to the choice (3.8) and is nilpotent due to the factorization (3.19).

IV. UNITARY TRANSFORMATION AND CONSTRAINT OPERATORS

The former results can be generalized, whenever one takes into account transformations leaving the BRST system invariant. In fact, the essential properties of $\hat{\Omega}$ (Hermiticity and nilpotency) are not modified under a unitary transformation

$$\hat{\Omega} \rightarrow e^{i\hat{C}} \hat{\Omega} e^{-i\hat{C}}, \tag{4.1}$$

This transformation defines a new set of first class constraint operators. By choosing

$$\begin{aligned}
\hat{C} &= \frac{1}{2} [\hat{\eta}^1 F_1(q^i) \hat{\mathcal{P}}_1 - \hat{\mathcal{P}}_1 F_1(q^i) \hat{\eta}^1 + \hat{\eta}^2 F_2(q^j) \hat{\mathcal{P}}_2 - \hat{\mathcal{P}}_2 F_2(q^j) \hat{\eta}^2] \\
&= \hat{\eta}^1 F_1(q^i) \hat{\mathcal{P}}_1 + \hat{\eta}^2 F_2(q^j) \hat{\mathcal{P}}_2 + \frac{i}{2} F_1(q^i) + \frac{i}{2} F_2(q^i) \\
&= -\hat{\mathcal{P}}_1 F_1(q^i) \hat{\eta}^1 - \hat{\mathcal{P}}_2 F_2(q^j) \hat{\eta}^2 - \frac{i}{2} F_1(q^i) - \frac{i}{2} F_2(q^i), \quad F_{1,2}(q) > 0,
\end{aligned} \tag{4.2}$$

and using the identities

$$e^{i\hat{\eta}^A F_A(q^i) \hat{\mathcal{P}}_A} = 1 + i\hat{\eta}^A (e^{F_A(q^i)} - 1) \hat{\mathcal{P}}_A, \tag{4.3}$$

$$e^{i\hat{\mathcal{P}}_A F_A(q^i) \hat{\eta}^A} = 1 + i\hat{\mathcal{P}}_A (e^{F_A(q^i)} - 1) \hat{\eta}^A, \tag{4.4}$$

one obtains

$$\begin{aligned}
\hat{\Omega} = & \hat{\eta}^1 e^{\frac{F_1-F_2}{2}} \hat{\mathcal{H}}_1 e^{\frac{F_1+F_2}{2}} + \hat{\eta}^1 \hat{\eta}^2 i e^{\frac{F_1+F_2}{2}} [\hat{\mathcal{H}}_1, e^{-F_2}] e^{\frac{F_1+F_2}{2}} \hat{\mathcal{P}}_2 \\
& + \hat{\eta}^2 e^{\frac{F_2-F_1}{2}} \hat{\mathcal{H}}_2 e^{\frac{F_1+F_2}{2}} + \hat{\eta}^2 \hat{\eta}^1 i e^{\frac{F_1+F_2}{2}} [\hat{\mathcal{H}}_2, e^{-F_1}] e^{\frac{F_1+F_2}{2}} \hat{\mathcal{P}}_1 \\
& + \frac{1}{2} \hat{\eta}^1 \hat{\eta}^2 e^{F_1+F_2} \hat{c}_{12}^a \hat{\mathcal{P}}_a + \hat{\eta}^a e^{-\frac{F_1+F_2}{2}} \hat{\mathcal{H}}_a e^{\frac{F_1+F_2}{2}} \\
& + \hat{\eta}^1 \hat{\eta}^a \left(\hat{c}_{1a}^1 - i e^{\frac{F_1-F_2}{2}} [\hat{\mathcal{H}}_a, e^{-F_1}] e^{\frac{F_1+F_2}{2}} \right) \hat{\mathcal{P}}_1 \\
& + \hat{\eta}^2 \hat{\eta}^a \left(\hat{c}_{2a}^2 - i e^{\frac{-F_1+F_2}{2}} [\hat{\mathcal{H}}_a, e^{-F_2}] e^{\frac{F_1+F_2}{2}} \right) \hat{\mathcal{P}}_2 + \frac{1}{2} \hat{\eta}^a \hat{\eta}^b C_{ab}^c \hat{\mathcal{P}}_c
\end{aligned} \tag{4.5}$$

Thus one can identify constraint operators and structure functions in Eq. (4.5),

$$\hat{H}_1 = \frac{1}{2} e^{\frac{F_1-F_2}{2}} f_1^{-\frac{1}{2}} \hat{p}_{i_1} g^{i_1 j_1} f_1 \hat{p}_{j_1} f_1^{-\frac{1}{2}} e^{\frac{F_1+F_2}{2}} + e^{F_1} v_1, \tag{4.6}$$

$$\hat{H}_2 = \frac{1}{2} e^{\frac{F_2-F_1}{2}} f_2^{-\frac{1}{2}} \hat{p}_{i_2} g^{i_2 j_2} f_2 \hat{p}_{j_2} f_2^{-\frac{1}{2}} e^{\frac{F_1+F_2}{2}} + e^{F_2} v_2, \tag{4.7}$$

$$\hat{\mathcal{H}}_a = e^{-\frac{F_1+F_2}{2}} f^{\frac{1}{2}} \xi_a^i \hat{p}_i f^{-\frac{1}{2}} e^{\frac{F_1+F_2}{2}}. \tag{4.8}$$

The resulting constraint operators, Eqs. (4.6),(4.7), correspond to scaled super-Hamiltonian constraints $H_A = e^{F_A} \mathcal{H}_A$. One can name $G^{i_A j_A} = e^{F_A} g^{i_A j_A}$, $V_A = e^{F_A} v_A$ to write

$$\hat{H}_1 = \frac{1}{2} e^{\frac{F_1-F_2}{2}} f_1^{-\frac{1}{2}} \hat{p}_{i_1} e^{-F_1} G^{i_1 j_1} f_1 \hat{p}_{j_1} f_1^{-\frac{1}{2}} e^{\frac{F_1+F_2}{2}} + V_1, \tag{4.9}$$

$$\hat{H}_2 = \frac{1}{2} e^{\frac{F_2-F_1}{2}} f_2^{-\frac{1}{2}} \hat{p}_{i_2} e^{-F_2} G^{i_2 j_2} f_2 \hat{p}_{j_2} f_2^{-\frac{1}{2}} e^{\frac{F_1+F_2}{2}} + V_2, \tag{4.10}$$

with the corresponding structure functions

$$\begin{aligned}
\hat{C}_{AB}^B &= i e^{\frac{F_1+F_2}{2}} [\hat{\mathcal{H}}_A, e^{-F_B}] e^{\frac{F_1+F_2}{2}} \\
&= \frac{1}{2} e^{\frac{F_A}{2}} f^{-\frac{1}{2}} [\hat{p}_{i_A} f g^{i_A j_A} F_{B,j_A} + F_{B,j_A} f g^{i_A j_A} \hat{p}_{j_A}] f^{-\frac{1}{2}} e^{\frac{F_A}{2}},
\end{aligned} \tag{4.11}$$

$$\hat{C}_{12}^a = e^{\frac{F_1+F_2}{2}} \hat{c}_{12}^a, \tag{4.12}$$

$$\hat{C}_{Aa}^A = \hat{c}_{Aa}^A + \xi_a^i (F_A)_{,i}, \tag{4.13}$$

$$\hat{C}_{ab}^c = \hat{C}_{ab}^c. \tag{4.14}$$

All the operators and structure functions are ordered in such a way they satisfy,

$$[\hat{H}_1, \hat{H}_2] = \hat{C}_{12}^1(q, p)\hat{H}_1 + \hat{C}_{12}^2(q, p)\hat{H}_2 + \hat{C}_{12}^a\hat{\mathcal{H}}_a, \quad (4.15)$$

$$[\hat{H}_A, \hat{\mathcal{H}}_a] = \hat{C}_{Aa}^{(A)}\hat{H}_{(A)}, \quad (4.16)$$

$$[\hat{\mathcal{H}}_a, \hat{\mathcal{H}}_b] = \hat{C}_{ab}^c\hat{\mathcal{H}}_c \quad (4.17)$$

(there is no sum over label A), i.e., the algebra is free from anomalies at the quantum level.

Of course, the scaling of constraints does not modify the dynamics of the system. However the constraint algebra looks different: the Eq. (4.16) now involves all constraints and the structure functions are no longer constant. Since the only consequence of the proposed unitary transformation is the scaling of the super-Hamiltonians, the commutators among supermomenta remain unchanged.

V. PHYSICAL INNER PRODUCT

To complete the quantization it is necessary to define a physical inner product where the spurious degrees of freedom are frozen by means of gauge fixing conditions (see also Ref. [15]). Here it is relevant to take into account the role played by the invariance transformations of the theory: (i) general coordinate transformation, (ii) linear combinations of supermomenta constraints, (iii) scaling of super-Hamiltonian constraints. The physical inner product for Dirac wave functions

$$(\varphi_1, \varphi_2) = \int dq \left[\prod_{\chi}^{m+2} \delta(\chi) \right] J \varphi_1^*(q) \varphi_2(q) \quad (5.1)$$

(where J is the Faddeev-Popov determinant and χ represents $m + 2$ gauge conditions) must be invariant under any of these transformations. By regarding the behavior of the constraint operators under transformations (i-iii) one realizes that the Dirac wave functions should transform according to [5]

$$\varphi \rightarrow \varphi' = (\det A)^{\frac{1}{2}} e^{-\frac{F_1+F_2}{2}} \varphi, \quad (5.2)$$

A being the matrix of the combination of linear constraints. Therefore the Faddeev-Popov determinant J in the physical inner product should change in opposite way, in order that the inner product remains unchanged.

In Eq. (5.1) there are m functions χ fixing m coordinates associated to the supermomenta constraints, whose characteristics are very well known. So, let us pay attention to the remaining two gauge fixing functions, which come from the super-Hamiltonians and are involved with the scaling transformation. As it was shown in previous works, the scaling factor can be associated either with a positive definite potential (intrinsic time case [5]) or with the norm of a conformal Killing vector (extrinsic time case [6]). Up to this point it has not been necessary to make any assumption on the potentials of the super-Hamiltonians (except for their functional dependence). However, some care about the type of potential must be taken to fix the gauge. If the potentials are definite positive, the time is intrinsic and the scaling factor of the super-Hamiltonians will be the potentials themselves: $F_A = \ln V_A$ and the associated gauge condition will be the one studied in Ref. [5]. If the time is extrinsic, the scale factors are associated with the norm of a conformal Killing vector for each super-Hamiltonian: $F_A = \ln |\vec{\xi}_A|^{-2}$ and the inner product will be the one defined in Ref. [6].

However, there are $m + 2$ constraints in the theory, so we need one more gauge condition. Since, this finite dimensional model mimics the constraint algebra of general relativity, the two subspaces (q^{i_1}) and (q^{i_2}) are interpreted as the field at two different points of the space-time, and so the super-Hamiltonians \mathcal{H}_1 and \mathcal{H}_2 are regarded as “the super-Hamiltonian” evaluated in two different points in space-time. Then, we suppose that both super-Hamiltonian have the same type of time, and in fact, both contain the same time. So, we propose as the remaining gauge condition

$$\chi = \delta(t_1 - t_2). \quad (5.3)$$

This gauge condition can be naturally retrieved in the framework of the multiple time

formalism [9–11]. Therefore, in both cases -intrinsic and extrinsic time- the inner product is regularized by including Eq. (5.3) in the already known set of $m + 1$ gauge conditions.

VI. CONCLUSIONS

In this work we managed to naturally extend the previous ordering findings of Refs. [5–7] to the interesting case of systems subject to more than one super-Hamiltonian constraint. As long as we know, this is the first work that treat ordering problems in such case and it also prove the power of the BRST formalism for providing the necessary tools for the treatment of invariance properties at the quantum level. By taking advantage of these invariance transformations of the theory, in particular the scaling invariance of the super-Hamiltonians, we were able to raise a nontrivial algebra between the super-Hamiltonians and to find the anomaly-free ordering, namely Eqs. (4.15)-(4.17). In this case, each scaling independently contribute with a term, involving also the momenta constraints. The role played by the invariance transformations, and how they modify the operator ordering, has been briefly discussed in Sec. V (see also Refs. [5,6] for a more in depth analysis).

When a finite dimensional system is proposed as a model for quantizing a general covariant theory (such as for example, general relativity), always remains the suspicion about the real competence of the model when it is compared with the infinite dimensional case. However, we can learn something about the ordering of the constraint operators that must be applied to both cases: the invariance transformation of the theory substantially modify the ordering. For example, if one admits that the super-Hamiltonian in the infinite dimensional case can be scaled, then the ordering for the supermomenta cannot be a simple functional derivation with respect to the canonical field variable, as usually appears in the literature, but it must “wear” the scaling factors in both sides of it.

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